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PROMOTING MULTIPLE REPRESENTATIONS IN ALGEBRA

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Background

Many teachers and researchers know, that the presentation of algebra almost exclusively as the study of expressions and equations, can pose serious obstacles in the process of effective and meaningful learning (Kieran 1992). As a result, from the very beginning of learning algebra, the use of various representations is recommended (NCTM 1989).

The use of verbal, numerical, graphical, and algebraic representations has the potential of making the process of learning algebra meaningful and effective. In order that this potential be realized in practice, we must be aware of both the advantages and disadvantages of each representation. The following are some of the strengths and weaknesses of each representation.

a) *The verbal representation* is usually used in posing a problem, and is also needed in the final interpretation of the results obtained in the solution process. The verbal presentation of a problem creates a natural environment for understanding its context and for communicating its solution. Verbal reasoning can also be a powerful tool for solving problems and can facilitate the presentation and application of general patterns. It also emphasizes the connection between mathematics and other domains of academic and everyday life. On the other hand, the use of verbal language can be ambiguous and elicit irrelevant or misleading associations; it is less universal and its dependence on personal style can be an obstacle in mathematical communication. Moreover, if symbols are avoided, verbal communication or reasoning may be a less powerful mathematical tool than more formal approaches.

b) *The numerical representation* is familiar to students at the beginning algebra stage. Numerical approaches provide a convenient and effective bridge to algebra, and frequently precede the use of any other representation. The use of numbers is important in acquiring a first understanding of a problem, and in the investigation of particular cases. However, its lack of generality can be a disadvantage. A numerical approach may not be very effective in providing a general picture and as a result, some important aspects or solutions of a problem may be missed. Thus, its potential as a tool for solving problems may be sometimes quite limited.

c) *The graphical representation* is effective in providing a clear picture of a real valued function of a real variable. Graphs are intuitive and particularly appealing to students who like a visual approach. On the other hand, graphical representation may lack the required accuracy, is influenced by external factors (such as scaling) and frequently presents only a section of the problem's domain or range. Its power as a mathematical tool varies, according to the task at hand.

d) *The algebraic representation* is concise, general, and effective in the presentation of patterns and mathematical models and hence, it is a powerful tool. The manipulation of algebraic objects is sometimes the only method of justifying or proving general statements. However, an exclusive use of algebraic symbols (at *any* stage of learning) may blur or obstruct the mathematical meaning or nature of the represented objects, and cause difficulties in some students' interpretation of their results.

The importance of working with various representations is a result of these and other advantages and disadvantages of each representation, and the need to cater to students' individual styles of thinking. Thus, both curriculum developers and teachers should be aware of the need to work in an environment of multiple representations – i.e., an environment that allows the representation of a problem and its solution in several ways (usually some, or all of the four representations mentioned above). Although each representation has its disadvantages, their combined use can cancel them out and proves to be an effective tool (Kaput 1992). The *Standards* (NCTM 1989) relate to this issue as following:

> *Different representations of problems serve as different lenses through which students interpret the problems and the solutions. If students are to become mathematically powerful, they must be flexible enough to approach situations in a variety of ways and recognize the relationships among different points of view.* (p. 84)

More specifically, Ainsworth and colleagues (1998) mention three reasons why multiple representations may promote learning: (a) it is highly probable that different representations express different aspects more clearly and that, hence, the information gained from combining representations will be greater than can be gained from a single representation; (b) multiple representations constrain each other, so that the space of permissible operators becomes smaller; (c) when required to relate multiple representations to each other, the learner has to engage in activities which promote understanding, in contrast to rote learning.

In the case of algebra learning, the use of computers contributes considerably to the promotion of multiple representations (Heid 1995). While working with spreadsheets and graph plotters, algebraic expressions become a natural requirement and provide an effective means for obtaining a numerical and graphical representation of the relevant data. In a learning environment that lacks computers, the drawing of graphs or the production of extended lists of numbers tends to be tedious and unrewarding.

In the process of solving a problem it is difficult to isolate representations. Thus in most cases, any approach is accompanied by verbal explanations or by numerical computations. In this paper we restrict ourselves to the use of representations as *mathematical tools* (and less as means of communication) in the context of beginning algebra. Thus the use of a sequence or table to answer a question will be an example of a numerical approach, whereas the use of verbal reasoning (possibly including some computations and numbers) will be considered a verbal approach. The use of graphs or algebraic expressions is easier to define and detect.

The ability to work with a variety of representations cannot be expected to develop spontaneously. Therefore, when learning algebra in either a technologically based or in a more conventional environment, student awareness and ability to use various representations must be promoted actively and systematically. We describe some ways in which tasks can be designed to promote the use of multiple representations. The following section presents, as an example, an activity taken from a beginning algebra course for seventh grade students, and discusses its potential to achieve this goal. We also report some findings about students' use of representations in an assessment task given at the end of one week of work on the activity.

Designing Tasks

In our attempts to promote student thinking and actions in a variety of representations, we found some effective types of tasks and questioning. We will analyze the structure of an activity called *Savings*, to illustrate the claim that tasks can be designed to encourage simultaneous use of several representations.

Describing the problem situation. Questions, tasks or even more complex activities are usually presented in one representation and they may, or may not, require the solver to make a transition to another representation. For example, students may decide to solve a verbally posed problem graphically or algebraically. (In some classrooms, the use of verbal reasoning to solve an algebra problem has not yet received full legitimization...)

We found that the presentation of various parts of a problem situation in different representations, encourages flexibility in students' choice of representations in their solution path, and increases their awareness of their solution style. The presentation of a problem in several representations gives legitimization to their use in the solution process. Moreover, in order to understand and solve such a problem, most students perform frequent transitions between representations, and perceive them as a natural need, rather than an arbitrary requirement. Posing a problem in several representations is particularly suitable for situations that require the parallel investigation of several methods, quantities etc. Figure 1 presents the *Savings* problem situation. In this activity, students investigate the weekly changes in the savings of four children, where the savings of each child are presented in a different representation.

Insert Figure 1 about here.

Posing investigative questions. The presentation of the problem situation is followed by a variety of questions, aimed at leading students through their. These questions are posed for a variety of reasons. The following categories of tasks are examples of possibilities to design activities that relate to our agenda of multiple representations. We will illustrate each category by a sample of questions from the *Savings* activity.

a) <u>Getting acquainted with the initial representation</u>. The first questions require students to analyze each component in its original presentation, make some extrapolations or draw some conclusions. At this stage, although not specifically required to do so, many students avoid any transitions from one representation to another. In our case, for example, we posed questions about the savings of each child. First, we asked for the amount of money at the end of a week specifically included in the data, then to extrapolate to a more advanced week, and finally, we asked for the week corresponding to a given amount of money.

b) Explicit requests for transitions between representations. At the next stage, students can be required to work in a specific representation. The following two, questions illustrate this stage.

> *Describe in words how the savings of each child change through the year.*
>
> *Given the graphs of the savings of all four children throughout the year, identify each graph and find the meaning and the value of each intersection point.*

c) Exploratory questions. Finally, students can be asked more complex and open-ended questions. At this stage, they are expected to choose their own method of representation and solution path. In our activity, we asked the students to compare the savings of the four children.

> *Compare the savings of two out of the four children.*
>
> *Use words like*   *the savings increase (or decrease)*
> *the savings increase or decrease at a rate of ...*
> *who has a larger (or smaller) amount at the beginning (or end)*
> *larger (or smaller) by … , double ... , equal.*
>
> *Use tables, graphs, expressions and explanations.*
>
> *Add another child to your comparison.*

Posing reflective questions. Awareness of the possibility to use various representations, of their advantages and disadvantages, of the student's personal style of thinking and of various ways to present the solution of a problem, are all important aspects of reflection. Reflective questions allow students to distance themselves from actions undertaken previously, and hence lead them to evaluate their own, and others' actions. Moreover, the ability to reflect on the solution of a problem increases considerably the solver's mathematical power (Hershkowitz and Schwarz 1999). As in the use of multiple representations, we cannot rely exclusively on a spontaneous development of the ability to reflect. The following types of tasks are examples of possibilities to design activities that make reflection an integral part of the solution process.

a) Description of work. The requirement to describe one's work is attached to many questions. This "habit" is more than routine, and its importance is beyond the need to document the solution. It allows students to reevaluate their solution strategies and eventually consider other possibilities. Sometimes, the task can be more specific. Thus, we can attach to the text of a problem a blank page called *Work Area*, with the words *Tables, Graphs, Expressions* and *Descriptions* on various parts of the page. In this case, the use of any particular representation is recommended, but optional. At other times, we can ask students directly to mention the representation they used on each occasion.

b) Commenting on others' work. Presenting one, or several (fictional) students' work, reduces the burden of getting involved in the actual process of solving a problem and allows students to relate to, and reflect on, particular aspects of the solution. Here is an example:

Ran wanted to find how much Dina had saved by the end of the 15th week. Vered suggested continuing the table a little more.

Week #	*11*	*12*	*13*	*14*	*15*
Amount	*77*	*84*	*91*	*98*	*105*

She looked at the table and found that the amount is $105.
Motty claimed that he had another way. Since Dina had no savings at the beginning of the year and her savings increased by 7 each week, she would have 7 times the number of weeks – that is $7 \cdot 15 = 105$.
Do you think that both methods are correct? Which method do you prefer?

c) Asking students to design their own questions. Another possible way to raise awareness of the potential of various representations, is to give a problem situation in one, or in multiple representations (possibly collected from students' previous work), and ask students to design (and solve) a question, which in their view, can be answered using the given representation. In the case of our activity, for example, we can pose this task, and enclose the tables, graphs, expressions or verbal descriptions of the savings of *all* four children.

d) Asking for reflection on mathematical concept. Journal items are particularly appropriate for asking students to reflect on possible ways to answer the posed questions, and of describing their solution. Thus, towards the end of our activity, the students were required to construct a concept map on means of representing data and solutions. They were also encouraged to discuss the advantages and disadvantages of using a particular representation.

Five)Allowing time for reflection. The solution of complex problems over a longer period of time (in our case, five lessons spread through one week) creates, of itself, further opportunities for spontaneous or induced reflection.

In the next section, we consider the use of various representations in the solution of a task by two classes of beginning algebra students, who worked on *Savings* and other similar activities.

Assessing Students' Use of Representations.

After a week of investigating the *Savings* activity (including one lesson of work with Excel), the teachers of two seventh grade beginning algebra classes gave an assessment task related to the same context. The task was given, about two months after the beginning of the course, to seventy students (M. T. was one of the teachers) who worked in (thirty five) pairs and without computers. Although the assessment of the students' work had a wider scope, we will present here only some findings that relate to their use of representations. At the initial stage of the task, the savings of two children during a year were described in a table and a graph. Then, the students `were required to answer a sequence of questions and were specifically instructed, both orally and in writing, to show their work and to mention the representation they used in each answer. Figure 2 presents the first seven (of ten) questions in this task.

Insert Figure 2 about here.

Algebraic expressions were mentioned at the beginning of the task as one of the four possible representations, but were not actually given. Our impression from the students' classroom work indicated that at this stage of the course, they preferred numerical or verbal solutions, and made a more limited use of graphs, and even less of algebraic expressions. We wanted, however, to have a more detailed picture of students' preferences and flexibility in their choice of representations. According to our (expert) view, different questions in the task favor different representations. Thus, the first two questions clearly favor the use of the given table of numbers, whereas in order to find the largest difference between their savings (Question 4), the use of graphs is more advantageous. In our opinion, the other questions could be answered with a reasonable investment of effort by choosing from several possible representations.

As expected, the use of the table of numbers was dominant. However, each question attracted various representations. Thus, in order to find the savings after half year (Question 1), the majority made direct use of the given table, as in the following solution:

> *We looked at Danny and Moshon's table and found the savings for the 26th week because 52:2 = 26*
>
> *$\frac{52}{2}$ → weeks in a year → divided by 2 for half a year*
>
> *Moshon's savings in the 26th week are $169 and Danny's savings are $170.*

The use of the table in order to answer Question 1 seems natural and simple. The following two examples show, however, that even in this case, the approaches to the solution were quite varied. Some pairs of students preferred the use of algebraic expressions (first example) or graphs (second example).

> *After half year, Moshon had $160. During the first week, Moshon had $30 and each week he added $5. Therefore we made the expression 30 + 5x x is the number of weeks and in order to compute [the amount after] half year, the expression will be 30 + 5·26 and in order to calculate on a calculator we need [to keep] the order of operation. Thus, we found 5·26 = 130 and added 30 = 160.*

> *We marked the midpoint of the horizontal axis and drew a line upwards (until our line intersects Danny and Moshon's points.) From Danny and Moshon's points we drew a horizontal line to the vertical*

axis. We discovered that Danny had $175 by the middle of the year and Moshon had only $150.

Question 7 related to the possibility of Danny and Moshon reaching a common sum of $400 and attracted many verbal solutions, like the (not completely correct) reasoning in the following example.

> *Danny and Moshon will never get the walkie-talkie because when Danny will have $300 (his largest amount) Moshon will have only $30 and when Moshon will have $290 (his largest amount) Danny will have only $40 and therefore the largest amount that they can reach is $330.*

An analysis of the students' answers to all seven questions showed that students were remarkably flexible in their use of representations. Only five pairs were consistently numerical. All the others used two, three, or four representations (31, 37 and 12 percent respectively). Sometimes, a pair used more than one representation in order to answer a question. Such transfers between representations occurred usually when work with the initially chosen representation seemed too difficult or unrewarding. The following two answers to Question 3 (finding cases when the difference between the savings is $60) illustrate this.

> *We compared the expressions: Moshon 30 + 5x and Danny 300 - 5x and then... ah...then we switched to an unsystematic and silly search of each number in the table (the results: the 21st , and the 33rd week).*

In this case, the pair of students attempted to answer the question algebraically, but lack of knowledge of how to use this tool forced them to switch to the numerical approach, in spite of their awareness of its disadvantages. In the next example, another pair makes a transition from a numerical, to a graphical representation. The transition to graphs is made, when the students' initial attempt provides an incomplete answer.

> *On the 21st week Danny had $60 more than Moshon. On the 33rd week Moshon had $60 more than Danny did. We looked for a difference of less than $100 and when we found them, we looked in Moshon's column when do we have to add or subtract $60 to get Danny's amount in the same week. We found in Danny's column $195*

and in the same week we found there in Moshon's column $135. Then we saw in the graph that the same case happens, only that [the amount of] Moshon is larger by $60 than Danny's amount.

Table 1 presents the distribution of the students' choice of representations on each of the first seven questions. Besides the obvious dominance of the numerical representation, it should be noted that some questions attracted a relatively large proportion of other representations. Thus, about 20 percent of the answers to Question 5 (finding when the savings are equal) were based on graphs and more than half of the answers to Question 7 (finding when the total savings exceed $400) were either verbal or algebraic.

Insert Table 1 about here.

Conclusion

Many mathematics educators recommend the use of multiple representations in algebra. We have tried to illustrate here some concrete ways of enhancing students' awareness of these advantages and their ability to use them in their routine work. The design of the *Savings* activity helped us to illustrate our belief that the promotion of multiple representations depends in the first place, on the presentation of a problem situation and on the nature of the questions asked. These should suggest, legitimize, recommend and sometimes even require the use of more than one representation. In order to internalize the principle of multiple representations, student reflection on these actions is also needed and should be consequently promoted by the task design.

The *Savings* activity and its follow-up task were conducted as regular classroom activities and were not planned and carried out as research. However, the analysis of students' responses supports our claim, that suitable problem posing and questioning, and systematic encouragement of students' experimentation with various representations, can increase awareness and ability to use various representations in the solution of a problem.

The predominant use of the numerical representation was expected. We relate this preference to the students' early stage in their learning of algebra and to the fact that in many cases, the nature of a task makes the use of a numerical approach mathematically sound.

"The good news", however, is that if students are provided with an appropriate learning environment, they will be able and willing to employ a wide variety of solution tools and paths. In our analysis of students' work, we found that the choice of a representation can be the result of the task's nature, personal preference, the problem solver's thinking style or attempts to overcome difficulties encountered during the use of another representation. Frequently, the choice of representation is influenced by a combination of several factors. Thus, in order to answer a question, students may choose a representation, based on their analysis of the problem and on personal preference, and switch to another representation at a later stage, as a result of difficulties in the solution process.

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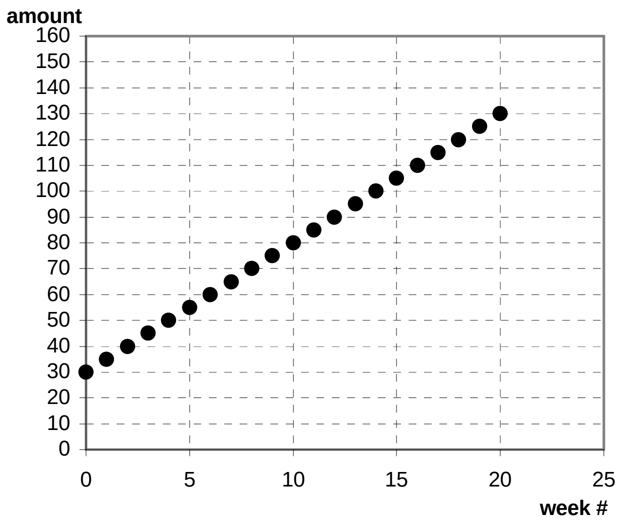
The savings of Dina, Yonni, Moshon and Danny changed during the last year, as described below. The numbers indicate amounts of money (in dollars) at the end of each week

Dina: The table shows how much money Dina had saved at the end of each week. The table continues in the same way for the rest of the year.

Week #	1	2	3	4	5	6	7	8	9...
Amount	7	14	21	28	35	42	49	56	63...

Yonni: Yonny kept his savings at $300 throughout the year.

Moshon: The graph describes Moshon's savings at the end of each of the first 20 weeks. The graph continues in the same way for the rest of the year.

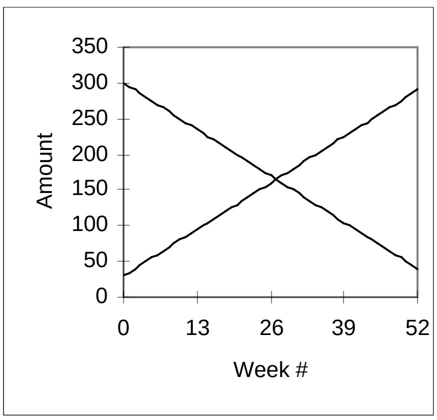


Danny: Danny's savings can be described by the expression 300 - 5x , where x stands for the number of weeks.

Figure 1. *Savings* – Problem situation.

The table and the graph below describe the savings of Danny and Moshon during the year.

Week #	Danny’s Savings	Moshon’s	Week #	Danny’s Savings	Moshon’s Savings
0	300	30	26	170	160
1	295	35	27	165	165
2	290	40	28	160	170
3	285	45	29	155	175
4	280	50	30	150	180
5	275	55	31	145	185
6	270	60	32	140	190
7	265	65	33	135	195
8	260	70	34	130	200
9	255	75	35	125	205
10	250	80	36	120	210
11	245	85	37	115	215
12	240	90	38	110	220
13	235	95	39	105	225
14	230	100	40	100	230
15	225	105	41	95	235
16	220	110	42	90	240
17	215	115	43	85	245
18	210	120	44	80	250
19	205	125	45	75	255
20	200	130	46	70	260
21	195	135	47	65	265
22	190	140	48	60	270
23	185	145	49	55	275
24	180	150	50	50	280
25	175	155	51	45	285
			52	40	290



<u>Figure2 (first part)</u>. *Savings* – Assessment task (the first seven questions).

Describe the work that you do to answer the questions below.

Please describe in detail **all** your (right or wrong) attempts, and the representations (table/expressions/graph/words) that you use to answer each question.

Important Remark:

The solution process is more important than the final result.

A detailed description of your work will improve your assessment.

The Questions:

1. How much had Moshon saved after half year? And how much did Danny have at the same time?
2. After how many weeks did each of the two children have $210?
3. When was difference between their savings $60? In whose favor was the difference?
4. Find the week with the largest difference between their savings.
5. Find the week when their savings were equal.
6. Find the week when the savings of one were double that of the other. In whose favor?
7. Danny and Moshon decided to pool their savings in order to buy a $400 walkie-talkie. Find the week in which they can realize their intention.

<u>Figure2 (Continued)</u>. *Savings* – Assessment task (the first seven questions).

Table 1. Choice of representation as percentage of total responses* for each question and the assessment task as a whole.

Question #	Numerical	Verbal	Graphical	Algebraic	Unidentifiable
1	68	0	5	16	11
2	74	3	5	13	5
3	71	10	7	7	5
4	51	14	19	0	16
5	61	5	24	0	10
6	59	13	13	2	13
7	25	41	8	13	13
Total	60	13	12	7	8

* The total number of students was 70 (35 pairs). However, if two representations were used in a pair's answer to a question, each representation was counted separately in the corresponding column.